

THETA SERIES ASSOCIATED WITH THE SCHRÖDINGER-WEIL REPRESENTATION

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ABSTRACT. In this paper, we define the Schrödinger-Weil representation for the Jacobi group and construct covariant maps for the Schrödinger-Weil representation. Using these covariant maps, we construct Jacobi forms with respect to an arithmetic subgroup of the Jacobi group.

1. Introduction

For a given fixed positive integer n , we let

$$\mathbb{H}_n = \{ \Omega \in \mathbb{C}^{(n,n)} \mid \Omega = {}^t\Omega, \quad \text{Im } \Omega > 0 \}$$

be the Siegel upper half plane of degree n and let

$$Sp(n, \mathbb{R}) = \{ g \in \mathbb{R}^{(2n,2n)} \mid {}^t g J_n g = J_n \}$$

be the symplectic group of degree n , where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F for two positive integers k and l , ${}^t M$ denotes the transposed matrix of a matrix M , $\text{Im } \Omega$ denotes the imaginary part of Ω and

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

We see that $Sp(n, \mathbb{R})$ acts on \mathbb{H}_n transitively by

$$g \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1},$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ and $\Omega \in \mathbb{H}_n$.

For two positive integers n and m , we consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \{ (\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{R}^{(m,n)}, \kappa \in \mathbb{R}^{(m,m)}, \kappa + \mu {}^t \lambda \text{ symmetric} \}$$

endowed with the following multiplication law

$$(\lambda, \mu; \kappa) \circ (\lambda', \mu'; \kappa') = (\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda {}^t \mu' - \mu {}^t \lambda').$$

We let

$$G^J = Sp(n, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)} \quad (\text{semi-direct product})$$

be the Jacobi group endowed with the following multiplication law

$$(g, (\lambda, \mu; \kappa)) \cdot (g', (\lambda', \mu'; \kappa')) = (gg', (\tilde{\lambda} + \lambda', \tilde{\mu} + \mu'; \kappa + \kappa' + \tilde{\lambda} {}^t \mu' - \tilde{\mu} {}^t \lambda'))$$

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with $g, g' \in Sp(n, \mathbb{R}), (\lambda, \mu; \kappa), (\lambda', \mu'; \kappa') \in H_{\mathbb{R}}^{(n,m)}$ and $(\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu)g'$. We let $\Gamma_n = Sp(n, \mathbb{Z})$ be the Siegel modular group of degree n . We let

$$\Gamma^J = \Gamma_n \ltimes H_{\mathbb{Z}}^{(n,m)}$$

be the Jacobi modular group. Then we have the *natural action* of G^J on the Siegel-Jacobi space $\mathbb{H}_{n,m} := \mathbb{H}_n \times \mathbb{C}^{(m,n)}$ defined by

$$(g, (\lambda, \mu; \kappa)) \cdot (\Omega, Z) = \left(g \cdot \Omega, (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1} \right),$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$, $(\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$. We refer to [19]-[25] for more details on materials related to the Siegel-Jacobi space.

The Weil representation for the symplectic group was first introduced by A. Weil in [13] to reformulate Siegel's analytic theory of quadratic forms (cf. [12]) in terms of the group theoretical theory. It is well known that the Weil representation plays a central role in the study of the transformation behaviors of the theta series. In this paper, we define the Schrödinger-Weil representation for the Jacobi group G^J . The aim of this paper is to construct the covariant maps for the Schrödinger-Weil representation, and to construct Jacobi forms with respect to an arithmetic subgroup of Γ^J using these covariant maps.

This paper is organized as follows. In Section 2, we discuss the Schrödinger representation of the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$ associated with a symmetric nonzero real matrix of degree m . In Section 3, we review the concept of a Jacobi form briefly. In Section 4, we define the Schrödinger-Weil representation $\omega_{\mathcal{M}}$ of the Jacobi group G^J associated with a symmetric positive definite matrix \mathcal{M} and provide some of the actions of $\omega_{\mathcal{M}}$ on the representation space $L^2(\mathbb{R}^{(m,n)})$ explicitly. In Section 5, we construct the covariant maps for the Schrödinger-Weil representation $\omega_{\mathcal{M}}$. In the final section we construct Jacobi forms with respect to an arithmetic subgroup of Γ^J using the covariant maps obtained in Section 5.

Notations : We denote by \mathbb{Z} and \mathbb{C} the ring of integers, and the field of complex numbers respectively. \mathbb{C}^{\times} denotes the multiplicative group of nonzero complex numbers. T denotes the multiplicative group of complex numbers of modulus one. The symbol “:=” means that the expression on the right is the definition of that on the left. For two positive integers k and l , $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F . For a square matrix $A \in F^{(k,k)}$ of degree k , $\sigma(A)$ denotes the trace of A . For any $M \in F^{(k,l)}$, ${}^t M$ denotes the transposed matrix of M . I_n denotes the identity matrix of degree n . We put $i = \sqrt{-1}$. For $z \in \mathbb{C}$, we define $z^{1/2} = \sqrt{z}$ so that $-\pi/2 < \arg(z^{1/2}) \leq \pi/2$. Further we put $z^{\kappa/2} = (z^{1/2})^{\kappa}$ for every $\kappa \in \mathbb{Z}$.

2. The Schrödinger Representation of $H_{\mathbb{R}}^{(n,m)}$

First of all, we observe that $H_{\mathbb{R}}^{(n,m)}$ is a 2-step nilpotent Lie group. The inverse of an element $(\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)}$ is given by

$$(\lambda, \mu; \kappa)^{-1} = (-\lambda, -\mu; -\kappa + \lambda {}^t \mu - \mu {}^t \lambda).$$

Now we set

$$[\lambda, \mu; \kappa] = (0, \mu; \kappa) \circ (\lambda, 0; 0) = (\lambda, \mu; \kappa - \mu^t \lambda).$$

Then $H_{\mathbb{R}}^{(n,m)}$ may be regarded as a group equipped with the following multiplication

$$[\lambda, \mu; \kappa] \diamond [\lambda_0, \mu_0; \kappa_0] = [\lambda + \lambda_0, \mu + \mu_0; \kappa + \kappa_0 + \lambda^t \mu_0 + \mu_0^t \lambda].$$

The inverse of $[\lambda, \mu; \kappa] \in H_{\mathbb{R}}^{(n,m)}$ is given by

$$[\lambda, \mu; \kappa]^{-1} = [-\lambda, -\mu; \kappa + \lambda^t \mu + \mu^t \lambda].$$

We set

$$L = \left\{ [0, \mu; \kappa] \in H_{\mathbb{R}}^{(n,m)} \mid \mu \in \mathbb{R}^{(m,n)}, \kappa = {}^t \kappa \in \mathbb{R}^{(m,m)} \right\}.$$

Then L is a commutative normal subgroup of $H_{\mathbb{R}}^{(n,m)}$. Let \widehat{L} be the Pontrajagin dual of L , i.e., the commutative group consisting of all unitary characters of L . Then \widehat{L} is isomorphic to the additive group $\mathbb{R}^{(m,n)} \times \text{Symm}(m, \mathbb{R})$ via

$$\langle a, \hat{a} \rangle = e^{2\pi i \sigma(\hat{\mu}^t \mu + \hat{\kappa} \kappa)}, \quad a = [0, \mu; \kappa] \in L, \hat{a} = (\hat{\mu}, \hat{\kappa}) \in \widehat{L},$$

where $\text{Symm}(m, \mathbb{R})$ denotes the space of all symmetric $m \times m$ real matrices.

We put

$$S = \left\{ [\lambda, 0; 0] \in H_{\mathbb{R}}^{(n,m)} \mid \lambda \in \mathbb{R}^{(m,n)} \right\} \cong \mathbb{R}^{(m,n)}.$$

Then S acts on L as follows:

$$\alpha_{\lambda}([0, \mu; \kappa]) = [0, \mu; \kappa + \lambda^t \mu + \mu^t \lambda], \quad [\lambda, 0, 0] \in S.$$

We see that the Heisenberg group $(H_{\mathbb{R}}^{(n,m)}, \diamond)$ is isomorphic to the semi-direct product $S \ltimes L$ of S and L whose multiplication is given by

$$(\lambda, a) \cdot (\lambda_0, a_0) = (\lambda + \lambda_0, a + \alpha_{\lambda}(a_0)), \quad \lambda, \lambda_0 \in S, a, a_0 \in L.$$

On the other hand, S acts on \widehat{L} by

$$\alpha_{\lambda}^*(\hat{a}) = (\hat{\mu} + 2\hat{\kappa}\lambda, \hat{\kappa}), \quad [\lambda, 0; 0] \in S, \quad a = (\hat{\mu}, \hat{\kappa}) \in \widehat{L}.$$

Then, we have the relation $\langle \alpha_{\lambda}(a), \hat{a} \rangle = \langle a, \alpha_{\lambda}^*(\hat{a}) \rangle$ for all $a \in L$ and $\hat{a} \in \widehat{L}$.

We have three types of S -orbits in \widehat{L} .

TYPE I. Let $\hat{\kappa} \in \text{Symm}(m, \mathbb{R})$ be nondegenerate. The S -orbit of $\hat{a}(\hat{\kappa}) = (0, \hat{\kappa}) \in \widehat{L}$ is given by

$$\widehat{\mathcal{O}}_{\hat{\kappa}} = \left\{ (2\hat{\kappa}\lambda, \hat{\kappa}) \in \widehat{L} \mid \lambda \in \mathbb{R}^{(m,n)} \right\} \cong \mathbb{R}^{(m,n)}.$$

TYPE II. Let $(\hat{\mu}, \hat{\kappa}) \in \mathbb{R}^{(m,n)} \times \text{Symm}(m, \mathbb{R})$ with degenerate $\hat{\kappa} \neq 0$. Then

$$\widehat{\mathcal{O}}_{(\hat{\mu}, \hat{\kappa})} = \left\{ (\hat{\mu} + 2\hat{\kappa}\lambda, \hat{\kappa}) \mid \lambda \in \mathbb{R}^{(m,n)} \right\} \subsetneq \mathbb{R}^{(m,n)} \times \{\hat{\kappa}\}.$$

TYPE III. Let $\hat{y} \in \mathbb{R}^{(m,n)}$. The S -orbit $\widehat{\mathcal{O}}_{\hat{y}}$ of $\hat{a}(\hat{y}) = (\hat{y}, 0)$ is given by

$$\widehat{\mathcal{O}}_{\hat{y}} = \{ (\hat{y}, 0) \} = \hat{a}(\hat{y}).$$

We have

$$\widehat{L} = \left(\bigcup_{\substack{\hat{\kappa} \in \text{Symm}(m, \mathbb{R}) \\ \hat{\kappa} \text{ nondegenerate}}} \widehat{\mathcal{O}}_{\hat{\kappa}} \right) \cup \left(\bigcup_{\hat{y} \in \mathbb{R}^{(m,n)}} \widehat{\mathcal{O}}_{\hat{y}} \right) \cup \left(\bigcup_{\substack{(\hat{\mu}, \hat{\kappa}) \in \mathbb{R}^{(m,n)} \times \text{Symm}(m, \mathbb{R}) \\ \hat{\kappa} \neq 0 \text{ degenerate}}} \widehat{\mathcal{O}}_{(\hat{\mu}, \hat{\kappa})} \right)$$

as a set. The stabilizer $S_{\hat{\kappa}}$ of S at $\hat{a}(\hat{\kappa}) = (0, \hat{\kappa})$ is given by

$$S_{\hat{\kappa}} = \{0\}.$$

And the stabilizer $S_{\hat{y}}$ of S at $\hat{a}(\hat{y}) = (\hat{y}, 0)$ is given by

$$S_{\hat{y}} = \left\{ [\lambda, 0; 0] \mid \lambda \in \mathbb{R}^{(m,n)} \right\} = S \cong \mathbb{R}^{(m,n)}.$$

In this section, for the present being we set $H = H_{\mathbb{R}}^{(n,m)}$ for brevity. We see that L is a closed, commutative normal subgroup of H . Since $(\lambda, \mu; \kappa) = (0, \mu; \kappa + \mu^t \lambda) \circ (\lambda, 0; 0)$ for $(\lambda, \mu; \kappa) \in H$, the homogeneous space $X = L \backslash H$ can be identified with $\mathbb{R}^{(m,n)}$ via

$$Lh = L \circ (\lambda, 0; 0) \longmapsto \lambda, \quad h = (\lambda, \mu; \kappa) \in H.$$

We observe that H acts on X by

$$(Lh) \cdot h_0 = L(\lambda + \lambda_0, 0; 0) = \lambda + \lambda_0,$$

where $h = (\lambda, \mu; \kappa) \in H$ and $h_0 = (\lambda_0, \mu_0; \kappa_0) \in H$.

If $h = (\lambda, \mu; \kappa) \in H$, we have

$$l_h = (0, \mu; \kappa + \mu^t \lambda), \quad s_h = (\lambda, 0; 0)$$

in the Mackey decomposition of $h = l_h \circ s_h$ (cf. [8]). Thus if $h_0 = (\lambda_0, \mu_0; \kappa_0) \in H$, then we have

$$s_h \circ h_0 = (\lambda, 0; 0) \circ (\lambda_0, \mu_0; \kappa_0) = (\lambda + \lambda_0, \mu_0; \kappa_0 + \lambda^t \mu_0)$$

and so

$$(2.1) \quad l_{s_h \circ h_0} = (0, \mu_0; \kappa_0 + \mu_0^t \lambda_0 + \lambda^t \mu_0 + \mu_0^t \lambda).$$

For a real symmetric matrix $c = {}^t c \in \text{Symm}(m, \mathbb{R})$ with $c \neq 0$, we consider the unitary character χ_c of L defined by

$$(2.2) \quad \chi_c((0, \mu; \kappa)) = e^{\pi i \sigma(c \kappa)} I, \quad (0, \mu; \kappa) \in L,$$

where I denotes the identity mapping. Then the representation $\mathcal{W}_c = \text{Ind}_L^H \chi_c$ of H induced from χ_c is realized on the Hilbert space $H(\chi_c) = L^2(X, d\lambda, \mathbb{C}) \cong L^2(\mathbb{R}^{(m,n)}, d\xi)$ as follows. If $h_0 = (\lambda_0, \mu_0; \kappa_0) \in H$ and $x = Lh \in X$ with $h = (\lambda, \mu; \kappa) \in H$, we have

$$(2.3) \quad (\mathcal{W}_c(h_0)f)(x) = \chi_c(l_{s_h \circ h_0})(f(xh_0)), \quad f \in H(\chi_c).$$

It follows from (2.1) that

$$(2.4) \quad (\mathcal{W}_c(h_0)f)(\lambda) = e^{\pi i \sigma\{c(\kappa_0 + \mu_0^t \lambda_0 + 2\lambda^t \mu_0)\}} f(\lambda + \lambda_0),$$

where $h_0 = (\lambda_0, \mu_0; \kappa_0) \in H$ and $\lambda \in \mathbb{R}^{(m,n)}$. Here we identified $x = Lh$ (resp. $xh_0 = Lh_0$) with λ (resp. $\lambda + \lambda_0$). The induced representation \mathcal{W}_c is called the *Schrödinger representation* of H associated with χ_c . Thus \mathcal{W}_c is a monomial representation.

Theorem 2.1. *Let c be a positive definite symmetric real matrix of degree m . Then the Schrödinger representation \mathcal{W}_c of H is irreducible.*

Proof. The proof can be found in [14], Theorem 3. \square

Remark. We refer to [14]-[18] for more representations of the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$ and their related topics.

3. Jacobi Forms

Let ρ be a rational representation of $GL(n, \mathbb{C})$ on a finite dimensional complex vector space V_{ρ} . Let $\mathcal{M} \in \mathbb{R}^{(m,m)}$ be a symmetric half-integral semi-positive definite matrix of degree m . Let $C^{\infty}(\mathbb{H}_{n,m}, V_{\rho})$ be the algebra of all C^{∞} functions on $\mathbb{H}_{n,m}$ with values in V_{ρ} . For $f \in C^{\infty}(\mathbb{H}_{n,m}, V_{\rho})$, we define

$$(3.1) \quad \begin{aligned} & (f|_{\rho, \mathcal{M}}[(g, (\lambda, \mu; \kappa))])(\Omega, Z) \\ &= e^{-2\pi i \sigma(\mathcal{M}(Z + \lambda\Omega + \mu)(C\Omega + D)^{-1}C^t(Z + \lambda\Omega + \mu))} \times e^{2\pi i \sigma(\mathcal{M}(\lambda\Omega^t\lambda + 2\lambda^tZ + \kappa + \mu^t\lambda))} \\ & \quad \times \rho(C\Omega + D)^{-1}f(g \cdot \Omega, (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1}), \end{aligned}$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$, $(\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$.

Definition 3.1. Let ρ and \mathcal{M} be as above. Let

$$H_{\mathbb{Z}}^{(n,m)} = \{(\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)} \mid \lambda, \mu \in \mathbb{Z}^{(m,n)}, \kappa \in \mathbb{Z}^{(m,m)}\}.$$

A *Jacobi form* of index \mathcal{M} with respect to ρ on a subgroup Γ of Γ_n of finite index is a holomorphic function $f \in C^{\infty}(\mathbb{H}_{n,m}, V_{\rho})$ satisfying the following conditions (A) and (B):

(A) $f|_{\rho, \mathcal{M}}[\tilde{\gamma}] = f$ for all $\tilde{\gamma} \in \Gamma \ltimes H_{\mathbb{Z}}^{(n,m)}$.

(B) For each $M \in \Gamma_n$, $f|_{\rho, \mathcal{M}}[M]$ has a Fourier expansion of the following form :

$$(f|_{\rho, \mathcal{M}}[M])(\Omega, Z) = \sum_{\substack{T=tT \geq 0 \\ \text{half-integral}}} \sum_{R \in \mathbb{Z}^{(n,m)}} c(T, R) \cdot e^{\frac{2\pi i}{\lambda_{\Gamma}} \sigma(T\Omega)} \cdot e^{2\pi i \sigma(RZ)}$$

with a suitable $\lambda_{\Gamma} \in \mathbb{Z}$ and $c(T, R) \neq 0$ only if $\begin{pmatrix} \frac{1}{\lambda_{\Gamma}}T & \frac{1}{2}R \\ \frac{1}{2}tR & \mathcal{M} \end{pmatrix} \geq 0$.

If $n \geq 2$, the condition (B) is superfluous by Koecher principle (cf. [26] Lemma 1.6). We denote by $J_{\rho, \mathcal{M}}(\Gamma)$ the vector space of all Jacobi forms of index \mathcal{M} with respect to ρ on Γ . Ziegler (cf. [26] Theorem 1.8 or [2] Theorem 1.1) proves that the vector space $J_{\rho, \mathcal{M}}(\Gamma)$ is finite dimensional. In the special case $\rho(A) = (\det(A))^k$ with $A \in GL(n, \mathbb{C})$ and a fixed $k \in \mathbb{Z}$, we write $J_{k, \mathcal{M}}(\Gamma)$ instead of $J_{\rho, \mathcal{M}}(\Gamma)$ and call k the *weight* of the corresponding Jacobi forms. For more results on Jacobi forms with $n > 1$ and $m > 1$, we refer to [19]-[22] and [26].

Definition 3.2. A Jacobi form $f \in J_{\rho, \mathcal{M}}(\Gamma)$ is said to be a *cusp* (or *cuspidal*) form if $\begin{pmatrix} \frac{1}{\lambda_{\Gamma}}T & \frac{1}{2}R \\ \frac{1}{2}tR & \mathcal{M} \end{pmatrix} > 0$ for any T, R with $c(T, R) \neq 0$. A Jacobi form $f \in J_{\rho, \mathcal{M}}(\Gamma)$ is said to

be *singular* if it admits a Fourier expansion such that a Fourier coefficient $c(T, R)$ vanishes unless $\det \begin{pmatrix} \frac{1}{\lambda_\Gamma} T & \frac{1}{2} R \\ \frac{1}{2} t R & \mathcal{M} \end{pmatrix} = 0$.

We allow a weight k to be half-integral.

Definition 3.3. Let $\Gamma \subset \Gamma_n$ be a subgroup of finite index. A holomorphic function $f : \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ is said to be a Jacobi form of a weight $k \in \frac{1}{2}\mathbb{Z}$ with level Γ and index \mathcal{M} if it satisfies the following transformation formula

$$(3.2) \quad f(\tilde{\gamma} \cdot (\Omega, Z)) = \chi(\tilde{\gamma}) J_{k,\mathcal{M}}(\tilde{\gamma}, (\Omega, Z)) f(\Omega, Z) \quad \text{for all } \tilde{\gamma} \in \tilde{\Gamma} = \Gamma \ltimes H_{\mathbb{Z}}^{(n,m)},$$

where χ is a character of $\tilde{\Gamma}$ and $J_{k,\mathcal{M}} : \tilde{\Gamma} \times \mathbb{H}_{n,m} \longrightarrow \mathbb{C}^\times$ is an automorphic factor defined by

$$J_{k,\mathcal{M}}(\tilde{\gamma}, (\Omega, Z)) = e^{2\pi i \sigma(\mathcal{M}(Z + \lambda\Omega + \mu)(C\Omega + D)^{-1}C^t(Z + \lambda\Omega + \mu))} \times e^{-2\pi i \sigma(\mathcal{M}(\lambda\Omega^t\lambda + 2\lambda^tZ + \kappa + \mu^t\lambda))} \det(C\Omega + D)^k$$

with $\tilde{\gamma} = (\gamma, (\lambda, \mu; \kappa)) \in \tilde{\Gamma}$ with $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$, $(\lambda, \mu; \kappa) \in H_{\mathbb{Z}}^{(n,m)}$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$.

4. The Schrödinger-Weil Representation

Throughout this section we assume that \mathcal{M} is a symmetric integral positive definite $m \times m$ matrix. We consider the Schrödinger representation $\mathcal{W}_{\mathcal{M}}$ of the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$ with the central character $\mathcal{W}_{\mathcal{M}}((0, 0; \kappa)) = \chi_{\mathcal{M}}((0, 0; \kappa)) = e^{\pi i \sigma(\mathcal{M}\kappa)}$, $\kappa \in \text{Symm}(m, \mathbb{R})$ (cf. (2.2)). We note that the symplectic group $Sp(n, \mathbb{R})$ acts on $H_{\mathbb{R}}^{(n,m)}$ by conjugation inside G^J . For a fixed element $g \in Sp(n, \mathbb{R})$, the irreducible unitary representation $\mathcal{W}_{\mathcal{M}}^g$ of $H_{\mathbb{R}}^{(n,m)}$ defined by

$$(4.1) \quad \mathcal{W}_{\mathcal{M}}^g(h) = \mathcal{W}_{\mathcal{M}}(ghg^{-1}), \quad h \in H_{\mathbb{R}}^{(n,m)}$$

has the property that

$$\mathcal{W}_{\mathcal{M}}^g((0, 0; \kappa)) = \mathcal{W}_{\mathcal{M}}((0, 0; \kappa)) = e^{\pi i \sigma(\mathcal{M}\kappa)} \text{Id}_{H(\chi_{\mathcal{M}})}, \quad \kappa \in \text{Symm}(m, \mathbb{R}).$$

Here $\text{Id}_{H(\chi_{\mathcal{M}})}$ denotes the identity operator on the Hilbert space $H(\chi_{\mathcal{M}})$. According to Stone-von Neumann theorem, there exists a unitary operator $R_{\mathcal{M}}(g)$ on $H(\chi_{\mathcal{M}})$ such that $R_{\mathcal{M}}(g)\mathcal{W}_{\mathcal{M}}(h) = \mathcal{W}_{\mathcal{M}}^g(h)R_{\mathcal{M}}(g)$ for all $h \in H_{\mathbb{R}}^{(n,m)}$. We observe that $R_{\mathcal{M}}(g)$ is determined uniquely up to a scalar of modulus one. From now on, for brevity, we put $G = Sp(n, \mathbb{R})$. According to Schur's lemma, we have a map $c_{\mathcal{M}} : G \times G \longrightarrow T$ satisfying the relation

$$R_{\mathcal{M}}(g_1g_2) = c_{\mathcal{M}}(g_1, g_2)R_{\mathcal{M}}(g_1)R_{\mathcal{M}}(g_2) \quad \text{for all } g_1, g_2 \in G.$$

Therefore $R_{\mathcal{M}}$ is a projective representation of G on $H(\chi_{\mathcal{M}})$ and $c_{\mathcal{M}}$ defines the cocycle class in $H^2(G, T)$. The cocycle $c_{\mathcal{M}}$ yields the central extension $G_{\mathcal{M}}$ of G by T . The group $G_{\mathcal{M}}$ is a set $G \times T$ equipped with the following multiplication

$$(g_1, t_1) \cdot (g_2, t_2) = (g_1g_2, t_1t_2 c_{\mathcal{M}}(g_1, g_2)^{-1}), \quad g_1, g_2 \in G, t_1, t_2 \in T.$$

We see immediately that the map $\tilde{R}_{\mathcal{M}} : G_{\mathcal{M}} \longrightarrow GL(H(\chi_{\mathcal{M}}))$ defined by

$$(4.2) \quad \tilde{R}_{\mathcal{M}}(g, t) = t R_{\mathcal{M}}(g) \quad \text{for all } (g, t) \in G_{\mathcal{M}}$$

is a true representation of $G_{\mathcal{M}}$. As in Section 1.7 in [7], we can define the map $s_{\mathcal{M}} : G \longrightarrow T$ satisfying the relation

$$c_{\mathcal{M}}(g_1, g_2)^2 = s_{\mathcal{M}}(g_1)^{-1} s_{\mathcal{M}}(g_2)^{-1} s_{\mathcal{M}}(g_1 g_2) \quad \text{for all } g_1, g_2 \in G.$$

Thus we see that

$$G_{2, \mathcal{M}} = \{ (g, t) \in G_{\mathcal{M}} \mid t^2 = s_{\mathcal{M}}(g)^{-1} \}$$

is the metaplectic group associated with \mathcal{M} that is a two-fold covering group of G . The restriction $R_{2, \mathcal{M}}$ of $\tilde{R}_{\mathcal{M}}$ to $G_{2, \mathcal{M}}$ is the Weil representation of G associated with \mathcal{M} . Now we define the projective representation $\pi_{\mathcal{M}}$ of the Jacobi group G^J by

$$(4.3) \quad \pi_{\mathcal{M}}(hg) = \mathcal{W}_{\mathcal{M}}(h) R_{\mathcal{M}}(g), \quad h \in H_{\mathbb{R}}^{(n, m)}, \quad g \in G.$$

The projective representation $\pi_{\mathcal{M}}$ of G^J is naturally extended to the true representation $\omega_{\mathcal{M}}$ of the group $G_{2, \mathcal{M}}^J = G_{2, \mathcal{M}} \ltimes H_{\mathbb{R}}^{(n, m)}$. The representation $\omega_{\mathcal{M}}$ is called the *Schrödinger-Weil representation* of G^J . Indeed we have

$$(4.4) \quad \omega_{\mathcal{M}}(h \cdot (g, t)) = t \mathcal{W}_{\mathcal{M}}(h) R_{\mathcal{M}}(g), \quad h \in H_{\mathbb{R}}^{(n, m)}, \quad (g, t) \in G_{2, \mathcal{M}}.$$

We recall that the following matrices

$$\begin{aligned} t_0(b) &= \begin{pmatrix} I_n & b \\ 0 & I_n \end{pmatrix} \text{ with any } b = {}^t b \in \mathbb{R}^{(n, n)}, \\ g_0(\alpha) &= \begin{pmatrix} {}^t \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \text{ with any } \alpha \in GL(n, \mathbb{R}), \\ \sigma_{n, 0} &= \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \end{aligned}$$

generate the symplectic group $G = Sp(n, \mathbb{R})$ (cf. [3, p.326], [10, p.210]). Therefore the following elements $h_t(\lambda, \mu; \kappa)$, $t_{\mathcal{M}}(b)$, $g_{\mathcal{M}}(\alpha)$ and $\sigma_{n, \mathcal{M}}$ of $G_{\mathcal{M}} \ltimes H_{\mathbb{R}}^{(n, m)}$ defined by

$$\begin{aligned} h_t(\lambda, \mu; \kappa) &= ((I_{2n}, t), (\lambda, \mu; \kappa)) \text{ with } t \in T, \lambda, \mu \in \mathbb{R}^{(m, n)} \text{ and } \kappa \in \mathbb{R}^{(m, m)}, \\ t_{\mathcal{M}}(b) &= ((t_0(b), 1), (0, 0; 0)) \text{ with any } b = {}^t b \in \mathbb{R}^{(n, n)}, \\ g_{\mathcal{M}}(\alpha) &= ((g_0(\alpha), 1), (0, 0; 0)) \text{ with any } \alpha \in GL(n, \mathbb{R}), \\ \sigma_{n, \mathcal{M}} &= ((\sigma_{n, 0}, 1), (0, 0; 0)), \end{aligned}$$

generate the group $G_{\mathcal{M}} \ltimes H_{\mathbb{R}}^{(n, m)}$. We can show that the representation $\tilde{R}_{\mathcal{M}}$ is realized on the representation $H(\chi_{\mathcal{M}}) = L^2(\mathbb{R}^{(m, n)})$ as follows: for each $f \in L^2(\mathbb{R}^{(m, n)})$ and $x \in \mathbb{R}^{(m, n)}$, the actions of $\tilde{R}_{\mathcal{M}}$ on the generators are given by

$$(4.5) \quad \left(\tilde{R}_{\mathcal{M}}(h_t(\lambda, \mu; \kappa)) f \right) (x) = t e^{\pi i \sigma \{ \mathcal{M}(\kappa + \mu {}^t \lambda + 2x {}^t \mu) \}} f(x + \lambda),$$

$$(4.6) \quad \left(\tilde{R}_{\mathcal{M}}(t_{\mathcal{M}}(b))f \right)(x) = e^{\pi i \sigma(\mathcal{M} x b^t x)} f(x),$$

$$(4.7) \quad \left(\tilde{R}_{\mathcal{M}}(g_{\mathcal{M}}(\alpha))f \right)(x) = (\det \alpha)^{\frac{m}{2}} f(x^t \alpha),$$

$$(4.8) \quad \left(\tilde{R}_{\mathcal{M}}(\sigma_{n,\mathcal{M}})f \right)(x) = \left(\frac{1}{i} \right)^{\frac{mn}{2}} (\det \mathcal{M})^{\frac{n}{2}} \int_{\mathbb{R}^{(m,n)}} f(y) e^{-2\pi i \sigma(\mathcal{M} y^t x)} dy.$$

We denote by $L_+^2(\mathbb{R}^{(m,n)})$ (resp. $L_-^2(\mathbb{R}^{(m,n)})$) the subspace of $L^2(\mathbb{R}^{(m,n)})$ consisting of even (resp. odd) functions in $L^2(\mathbb{R}^{(m,n)})$. According to Formulas (4.6)-(4.8), $R_{2,\mathcal{M}}$ is decomposed into representations of $R_{2,\mathcal{M}}^{\pm}$

$$R_{2,\mathcal{M}} = R_{2,\mathcal{M}}^+ \oplus R_{2,\mathcal{M}}^-,$$

where $R_{2,\mathcal{M}}^+$ and $R_{2,\mathcal{M}}^-$ are the even Weil representation and the odd Weil representation of G that are realized on $L_+^2(\mathbb{R}^{(m,n)})$ and $L_-^2(\mathbb{R}^{(m,n)})$ respectively. Obviously the center $\mathcal{Z}_{2,\mathcal{M}}^J$ of $G_{2,\mathcal{M}}^J$ is given by

$$\mathcal{Z}_{2,\mathcal{M}}^J = \{ ((I_{2n}, 1), (0, 0; \kappa)) \in G_{2,\mathcal{M}}^J \} \cong \text{Symm}(m, \mathbb{R}).$$

We note that the restriction of $\omega_{\mathcal{M}}$ to $G_{2,\mathcal{M}}$ coincides with $R_{2,\mathcal{M}}$ and $\omega_{\mathcal{M}}(h) = \mathcal{W}_{\mathcal{M}}(h)$ for all $h \in H_{\mathbb{R}}^{(n,m)}$.

Remark. In the case $n = m = 1$, $\omega_{\mathcal{M}}$ is dealt in [1] and [9]. We refer to [5] and [6] for more details about the Weil representation $R_{2,\mathcal{M}}$.

5. Covariant Maps for the Schrödinger-Weil representation

As before we let \mathcal{M} be a symmetric positive definite $m \times m$ real matrix. We define the mapping $\mathcal{F}^{(\mathcal{M})} : \mathbb{H}_{n,m} \longrightarrow L^2(\mathbb{R}^{(m,n)})$ by

$$(5.1) \quad \mathcal{F}^{(\mathcal{M})}(\Omega, Z)(x) = e^{\pi i \sigma\{\mathcal{M}(x \Omega^t x + 2x^t Z)\}}, \quad (\Omega, Z) \in \mathbb{H}_{n,m}, \quad x \in \mathbb{R}^{(m,n)}.$$

For brevity we put $\mathcal{F}_{\Omega, Z}^{(\mathcal{M})} := \mathcal{F}^{(\mathcal{M})}(\Omega, Z)$ for $(\Omega, Z) \in \mathbb{H}_{n,m}$.

We define the automorphic factor $J_{\mathcal{M}} : G^J \times \mathbb{H}_{n,m} \longrightarrow \mathbb{C}^{\times}$ for G^J on $\mathbb{H}_{n,m}$ by

$$(5.2) \quad J_{\mathcal{M}}(\tilde{g}, (\Omega, Z)) = e^{\pi i \sigma(\mathcal{M}(Z + \lambda \Omega + \mu)(C \Omega + D)^{-1} C^t (Z + \lambda \Omega + \mu))} \times e^{-\pi i \sigma(\mathcal{M}(\lambda \Omega^t \lambda + 2 \lambda^t Z + \kappa + \mu^t \lambda))} \det(C \Omega + D)^{\frac{m}{2}},$$

where $\tilde{g} = (g, (\lambda, \mu; \kappa)) \in G^J$ with $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$, $(\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$.

Theorem 5.1. *The map $\mathcal{F}^{(\mathcal{M})} : \mathbb{H}_{n,m} \longrightarrow L^2(\mathbb{R}^{(m,n)})$ defined by (5.1) is a covariant map for the Schrödinger-Weil representation $\omega_{\mathcal{M}}$ of G^J and the automorphic factor $J_{\mathcal{M}}$ for G^J on $\mathbb{H}_{n,m}$ defined by Formula (5.2). In other words, $\mathcal{F}^{(\mathcal{M})}$ satisfies the following covariance relation*

$$(5.3) \quad \omega_{\mathcal{M}}(\tilde{g}) \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} = J_{\mathcal{M}}(\tilde{g}, (\Omega, Z))^{-1} \mathcal{F}_{\tilde{g} \cdot (\Omega, Z)}^{(\mathcal{M})}$$

for all $\tilde{g} \in G^J$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$.

Proof. For an element $\tilde{g} = (g, (\lambda, \mu; \kappa)) \in G^J$ with $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$, we put $(\Omega_*, Z_*) = \tilde{g} \cdot (\Omega, Z)$ for $(\Omega, Z) \in \mathbb{H}_{n,m}$. Then we have

$$\begin{aligned} \Omega_* &= g \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \\ Z_* &= (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1}. \end{aligned}$$

In this section we use the notations $t_0(b)$, $g_0(\alpha)$ and $\sigma_{n,0}$ in Section 4. Since the following elements $h(\lambda, \mu; \kappa)$, $t(b)$, $g(\alpha)$ and σ_n of G^J defined by

$$\begin{aligned} h(\lambda, \mu; \kappa) &= (I_{2n}, (\lambda, \mu; \kappa)) \quad \text{with } \lambda, \mu \in \mathbb{R}^{(m,n)}, \kappa \in \mathbb{R}^{(m,m)}, \\ t(b) &= (t_0(b), (0, 0; 0)) \quad \text{with } b = {}^t b \in \mathbb{R}^{(m,m)}, \\ g(\alpha) &= (g_0(\alpha), (0, 0; 0)) \quad \text{with } \alpha \in GL(n, \mathbb{R}), \\ \sigma_n &= (\sigma_{n,0}, (0, 0; 0)) \end{aligned}$$

generate the Jacobi group, it suffices to prove the covariance relation (5.3) for the above generators.

Case I. $\tilde{g} = h(\lambda, \mu; \kappa)$ with $\lambda, \mu \in \mathbb{R}^{(m,n)}$, $\kappa \in \mathbb{R}^{(m,m)}$.

In this case, we have

$$\Omega_* = \Omega, \quad Z_* = Z + \lambda\Omega + \mu$$

and

$$J_{\mathcal{M}}(\tilde{g}, (\Omega, Z)) = e^{-\pi i \sigma \{ \mathcal{M}(\lambda \Omega {}^t \lambda + 2 \lambda {}^t Z + \kappa + \mu {}^t \lambda) \}}.$$

According to Formula (4.5), for $x \in \mathbb{R}^{(m,n)}$,

$$\begin{aligned} & \left(\omega_{\mathcal{M}}(h(\lambda, \mu; \kappa)) \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \right) (x) \\ &= e^{\pi i \sigma \{ \mathcal{M}(\kappa + \mu {}^t \lambda + 2 x {}^t \mu) \}} \mathcal{F}_{\Omega, Z}^{(\mathcal{M})}(x + \lambda) \\ &= e^{\pi i \sigma \{ \mathcal{M}(\kappa + \mu {}^t \lambda + 2 x {}^t \mu) \}} e^{\pi i \sigma \{ \mathcal{M}((x + \lambda)\Omega {}^t (x + \lambda) + 2 (x + \lambda) {}^t Z) \}}. \end{aligned}$$

On the other hand, according to Formula (5.2), for $x \in \mathbb{R}^{(m,n)}$,

$$\begin{aligned} & J_{\mathcal{M}}(h(\lambda, \mu; \kappa), (\Omega, Z))^{-1} \mathcal{F}_{\tilde{g} \cdot (\Omega, Z)}^{(\mathcal{M})}(x) \\ &= J_{\mathcal{M}}(h(\lambda, \mu; \kappa), (\Omega, Z))^{-1} \mathcal{F}_{\Omega, Z + \lambda\Omega + \mu}^{(\mathcal{M})}(x) \\ &= e^{\pi i \sigma \{ \mathcal{M}(\lambda \Omega {}^t \lambda + 2 \lambda {}^t Z + \kappa + \mu {}^t \lambda) \}} \cdot e^{\pi i \sigma \{ \mathcal{M}(x \Omega {}^t x + 2 x {}^t (Z + \lambda\Omega + \mu)) \}} \\ &= e^{\pi i \sigma \{ \mathcal{M}(\kappa + \mu {}^t \lambda + 2 x {}^t \mu) \}} e^{\pi i \sigma \{ \mathcal{M}((x + \lambda)\Omega {}^t (x + \lambda) + 2 (x + \lambda) {}^t Z) \}}. \end{aligned}$$

Therefore we prove the covariance relation (5.3) in the case $\tilde{g} = h(\lambda, \mu; \kappa)$ with λ, μ, κ real.

Case II. $\tilde{g} = t(b)$ with $b = {}^t b \in \mathbb{R}^{(n,n)}$.

In this case, we have

$$\Omega_* = \Omega + b, \quad Z_* = Z \quad \text{and} \quad J_{\mathcal{M}}(\tilde{g}, (\Omega, Z)) = 1.$$

According to Formula (4.6), we obtain

$$\left(\omega_{\mathcal{M}}(\tilde{g}) \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \right) (x) = e^{\pi i \sigma(\mathcal{M} x b^t x)} \mathcal{F}_{\Omega, Z}^{(\mathcal{M})}(x), \quad x \in \mathbb{R}^{(m,n)}.$$

On the other hand, according to Formula (5.2), for $x \in \mathbb{R}^{(m,n)}$, we obtain

$$\begin{aligned} & J_{\mathcal{M}}(\tilde{g}, (\Omega, Z))^{-1} \mathcal{F}_{\tilde{g} \cdot (\Omega, Z)}^{(\mathcal{M})}(x) \\ &= \mathcal{F}_{\Omega+b, Z}^{(\mathcal{M})}(x) \\ &= e^{\pi i \sigma(\mathcal{M}(x(\Omega+b)^t x + 2x^t Z))} \\ &= e^{\pi i \sigma(\mathcal{M} x b^t x)} \mathcal{F}_{\Omega, Z}^{(\mathcal{M})}(x). \end{aligned}$$

Therefore we prove the covariance relation (5.3) in the case $\tilde{g} = t(b)$ with $b = {}^t b \in \mathbb{R}^{(n,n)}$.

Case III. $\tilde{g} = g(\alpha)$ with $\alpha \in GL(n, \mathbb{R})$.

In this case, we have

$$\Omega_* = {}^t \alpha \Omega \alpha, \quad Z_* = Z \alpha$$

and

$$J_{\mathcal{M}}(\tilde{g}, (\Omega, Z)) = (\det \alpha)^{-\frac{m}{2}}.$$

According to Formula (4.7), for $x \in \mathbb{R}^{(m,n)}$,

$$\begin{aligned} & \left(\omega_{\mathcal{M}}(\tilde{g}) \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \right) (x) \\ &= (\det \alpha)^{\frac{m}{2}} \mathcal{F}_{\Omega, Z}^{(\mathcal{M})}(x^t \alpha) \\ &= (\det \alpha)^{\frac{m}{2}} \cdot e^{\pi i \sigma\{\mathcal{M}(x^t \alpha \Omega^t (x^t \alpha) + 2x^t \alpha^t Z)\}}. \end{aligned}$$

On the other hand, according to Formula (5.2), for $x \in \mathbb{R}^{(m,n)}$,

$$\begin{aligned} & J_{\mathcal{M}}(\tilde{g}, (\Omega, Z))^{-1} \mathcal{F}_{\tilde{g} \cdot (\Omega, Z)}^{(\mathcal{M})}(x) \\ &= (\det \alpha)^{\frac{m}{2}} \mathcal{F}_{t \alpha \Omega \alpha, Z \alpha}^{(\mathcal{M})}(x) \\ &= (\det \alpha)^{\frac{m}{2}} \cdot e^{\pi i \sigma\{\mathcal{M}(x^t \alpha \Omega^t (x^t \alpha) + 2x^t \alpha^t Z)\}}. \end{aligned}$$

Therefore we prove the covariance relation (5.3) in the case $\tilde{g} = g(\alpha)$ with $\alpha \in GL(n, \mathbb{R})$.

Case IV. $\tilde{g} = \left(\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, (0, 0; 0) \right)$.

In this case, we have

$$\Omega_* = -\Omega^{-1}, \quad Z_* = Z \Omega^{-1}$$

and

$$J_{\mathcal{M}}(\tilde{g}, (\Omega, Z)) = e^{\pi i \sigma(\mathcal{M} Z \Omega^{-1} {}^t Z)} (\det \Omega)^{\frac{m}{2}}.$$

In order to prove the covariance relation (5.3), we need the following useful lemma.

Lemma 5.1. *For a fixed element $\Omega \in \mathbb{H}_n$ and a fixed element $Z \in \mathbb{C}^{(m,n)}$, we obtain the following property*

$$(5.4) \quad \int_{\mathbb{R}^{(m,n)}} e^{\pi i \sigma(x \Omega^t x + 2x^t Z)} dx_{11} \cdots dx_{mn} = \left(\det \frac{\Omega}{i} \right)^{-\frac{m}{2}} e^{-\pi i \sigma(Z \Omega^{-1} t Z)},$$

where $x = (x_{ij}) \in \mathbb{R}^{(m,n)}$.

Proof of Lemma 5.1. By a simple computation, we see that

$$e^{\pi i \sigma(x \Omega^t x + 2x^t Z)} = e^{-\pi i \sigma(Z \Omega^{-1} t Z)} \cdot e^{\pi i \sigma\{(x + Z \Omega^{-1}) \Omega^t (x + Z \Omega^{-1})\}}.$$

Since the real Jacobi group $Sp(n, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(m,n)}$ acts on $\mathbb{H}_{n,m}$ holomorphically, we may put

$$\Omega = i A^t A, \quad Z = iV, \quad A \in \mathbb{R}^{(n,n)}, \quad V = (v_{ij}) \in \mathbb{R}^{(m,n)}.$$

Then we obtain

$$\begin{aligned} & \int_{\mathbb{R}^{(m,n)}} e^{\pi i \sigma(x \Omega^t x + 2x^t Z)} dx_{11} \cdots dx_{mn} \\ &= e^{-\pi i \sigma(Z \Omega^{-1} t Z)} \int_{\mathbb{R}^{(m,n)}} e^{\pi i \sigma\{(x + iV(A^t A)^{-1})(A^t A)^t (x + iV(A^t A)^{-1})\}} dx_{11} \cdots dx_{mn} \\ &= e^{-\pi i \sigma(Z \Omega^{-1} t Z)} \int_{\mathbb{R}^{(m,n)}} e^{\pi i \sigma\{(x + V(A^t A)^{-1}) A^t A^t (x + V(A^t A)^{-1})\}} dx_{11} \cdots dx_{mn} \\ &= e^{-\pi i \sigma(Z \Omega^{-1} t Z)} \int_{\mathbb{R}^{(m,n)}} e^{-\pi \sigma\{(uA)^t (uA)\}} du_{11} \cdots du_{mn} \quad (\text{Put } u = x + V(A^t A)^{-1} = (u_{ij})) \\ &= e^{-\pi i \sigma(Z \Omega^{-1} t Z)} \int_{\mathbb{R}^{(m,n)}} e^{-\pi \sigma(w^t w)} (\det A)^{-m} dw_{11} \cdots dw_{mn} \quad (\text{Put } w = uA = (w_{ij})) \\ &= e^{-\pi i \sigma(Z \Omega^{-1} t Z)} (\det A)^{-m} \cdot \left(\prod_{i=1}^m \prod_{j=1}^n \int_{\mathbb{R}} e^{-\pi w_{ij}^2} dw_{ij} \right) \\ &= e^{-\pi i \sigma(Z \Omega^{-1} t Z)} (\det A)^{-m} \quad (\text{because } \int_{\mathbb{R}} e^{-\pi w_{ij}^2} dw_{ij} = 1 \text{ for all } i, j) \\ &= e^{-\pi i \sigma(Z \Omega^{-1} t Z)} (\det(A^t A))^{-\frac{m}{2}} \\ &= e^{-\pi i \sigma(Z \Omega^{-1} t Z)} \left(\det \left(\frac{\Omega}{i} \right) \right)^{-\frac{m}{2}}. \end{aligned}$$

This completes the proof of Lemma 5.1. \square

According to Formula (4.8), for $x \in \mathbb{R}^{(m,n)}$, we obtain

$$\begin{aligned} & \left(\omega_{\mathcal{M}}(\tilde{g}) \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \right) (x) \\ &= \left(\frac{1}{i} \right)^{\frac{mn}{2}} (\det \mathcal{M})^{\frac{n}{2}} \int_{\mathbb{R}^{(m,n)}} \mathcal{F}_{\Omega, Z}^{(\mathcal{M})}(y) e^{-2\pi i \sigma(\mathcal{M} y^t x)} dy \\ &= \left(\frac{1}{i} \right)^{\frac{mn}{2}} (\det \mathcal{M})^{\frac{n}{2}} \int_{\mathbb{R}^{(m,n)}} e^{\pi i \sigma\{\mathcal{M}(y \Omega^t y + 2y^t Z)\}} e^{-2\pi i \sigma(\mathcal{M} y^t x)} dy \\ &= \left(\frac{1}{i} \right)^{\frac{mn}{2}} (\det \mathcal{M})^{\frac{n}{2}} \int_{\mathbb{R}^{(m,n)}} e^{\pi i \sigma\{\mathcal{M}(y \Omega^t y + 2y^t (Z - x))\}} dy. \end{aligned}$$

If we substitute $u = \mathcal{M}^{\frac{1}{2}} y$, then $du = (\det \mathcal{M})^{\frac{n}{2}} dy$. Therefore according to Lemma 5.1, we obtain

$$\begin{aligned}
& \left(\omega_{\mathcal{M}}(\tilde{g}) \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \right) (x) \\
&= \left(\frac{1}{i} \right)^{\frac{mn}{2}} (\det \mathcal{M})^{\frac{n}{2}} \int_{\mathbb{R}^{(m,n)}} e^{\pi i \sigma(u \Omega^t u + 2 \mathcal{M}^{1/2} u^t (Z - x))} (\det \mathcal{M})^{-\frac{n}{2}} du \\
&= \left(\frac{1}{i} \right)^{\frac{mn}{2}} \int_{\mathbb{R}^{(m,n)}} e^{\pi i \sigma(u \Omega^t u + 2 u^t (\mathcal{M}^{1/2} (Z - x)))} du \\
&= \left(\frac{1}{i} \right)^{\frac{mn}{2}} \left(\det \frac{\Omega}{i} \right)^{-\frac{m}{2}} e^{-\pi i \sigma \{ \mathcal{M}^{1/2} (Z - x) \Omega^{-1} u^t (Z - x) \mathcal{M}^{1/2} \}} \quad (\text{by Lemma 5.1}) \\
&= (\det \Omega)^{-\frac{m}{2}} e^{-\pi i \sigma(\mathcal{M}(Z - x) \Omega^{-1} u^t (Z - x))} \\
&= (\det \Omega)^{-\frac{m}{2}} e^{-\pi i \sigma(\mathcal{M}(Z \Omega^{-1} u^t Z + x \Omega^{-1} u^t x - 2 Z \Omega^{-1} u^t x))}.
\end{aligned}$$

On the other hand, according to Formula (5.2), for $x \in \mathbb{R}^{(m,n)}$,

$$\begin{aligned}
& J_{\mathcal{M}}(\tilde{g}, (\Omega, Z))^{-1} \mathcal{F}_{\tilde{g}, (\Omega, Z)}^{(\mathcal{M})}(x) \\
&= e^{-\pi i \sigma(\mathcal{M} Z \Omega^{-1} u^t Z)} (\det \Omega)^{-\frac{m}{2}} \mathcal{F}_{-\Omega^{-1}, Z \Omega^{-1}}^{(\mathcal{M})}(x) \\
&= (\det \Omega)^{-\frac{m}{2}} e^{-\pi i \sigma(\mathcal{M} Z \Omega^{-1} u^t Z)} e^{\pi i \sigma \{ \mathcal{M}(x(-\Omega^{-1}) u^t x + 2 x^t (Z \Omega^{-1})) \}} \\
&= (\det \Omega)^{-\frac{m}{2}} e^{-\pi i \sigma(\mathcal{M}(Z \Omega^{-1} u^t Z + x \Omega^{-1} u^t x - 2 Z \Omega^{-1} u^t x))}.
\end{aligned}$$

Therefore we prove the covariance relation (5.3) in the case $\tilde{g} = \sigma_n$. Since $J_{\mathcal{M}}$ is an automorphic factor for G^J on $\mathbb{H}_{n,m}$, we see that if the covariance relation (5.3) holds for two elements \tilde{g}_1, \tilde{g}_2 in G^J , then it holds for $\tilde{g}_1 \tilde{g}_2$. Finally we complete the proof. \square

6. Construction of Jacobi Forms

Let (π, V_{π}) be a unitary representation of G^J on the representation space V_{π} . We assume that (π, V_{π}) satisfies the following conditions (A) and (B):

(A) There exists a vector valued map

$$\mathcal{F} : \mathbb{H}_{n,m} \longrightarrow V_{\pi}, \quad (\Omega, Z) \mapsto \mathcal{F}_{\Omega, Z} := \mathcal{F}(\Omega, Z)$$

satisfying the following covariance relation

$$(6.1) \quad \pi(\tilde{\gamma}) \mathcal{F}_{\Omega, Z} = \psi(\tilde{\gamma}) J(\tilde{\gamma}, (\Omega, Z))^{-1} \mathcal{F}_{\tilde{\gamma}, (\Omega, Z)} \quad \text{for all } \tilde{\gamma} \in G^J, (\Omega, Z) \in \mathbb{H}_{n,m},$$

where ψ is a character of G^J and $J : G^J \times \mathbb{H}_{n,m} \longrightarrow GL(1, \mathbb{C})$ is a certain automorphic factor for G^J on $\mathbb{H}_{n,m}$.

(B) Let $\tilde{\Gamma}$ be an arithmetic subgroup of Γ^J . There exists a linear functional $\theta : V_{\pi} \longrightarrow \mathbb{C}$ which is semi-invariant under the action of $\tilde{\Gamma}$, in other words, for all $\tilde{\gamma} \in \tilde{\Gamma}$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$,

$$(6.2) \quad \langle \pi^*(\tilde{\gamma}) \theta, \mathcal{F}_{\Omega, Z} \rangle = \langle \theta, \pi(\tilde{\gamma})^{-1} \mathcal{F}_{\Omega, Z} \rangle = \chi(\tilde{\gamma}) \langle \theta, \mathcal{F}_{\Omega, Z} \rangle,$$

where π^* is the contragredient of π and $\chi : \tilde{\Gamma} \longrightarrow T$ is a unitary character of $\tilde{\Gamma}$.

Under the assumptions (A) and (B) on a unitary representation (π, V_π) , we define the function Θ on $\mathbb{H}_{n,m}$ by

$$(6.3) \quad \Theta(\Omega, Z) := \langle \theta, \mathcal{F}_{\Omega, Z} \rangle = \theta(\mathcal{F}_{\Omega, Z}), \quad (\Omega, Z) \in \mathbb{H}_{n,m}.$$

We now shall see that Θ is an automorphic form on $\mathbb{H}_{n,m}$ with respect to $\tilde{\Gamma}$ for the automorphic factor J .

Lemma 6.1. *Let (π, V_π) be a unitary representation of G^J satisfying the above assumptions (A) and (B). Then the function Θ on $\mathbb{H}_{n,m}$ defined by (6.3) satisfies the following modular transformation behavior*

$$(6.4) \quad \Theta(\tilde{\gamma} \cdot (\Omega, Z)) = \psi(\tilde{\gamma})^{-1} \chi(\tilde{\gamma})^{-1} J(\tilde{\gamma}, (\Omega, Z)) \Theta(\Omega, Z)$$

for all $\tilde{\gamma} \in \tilde{\Gamma}$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$.

Proof. For any $\tilde{\gamma} \in \tilde{\Gamma}$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$, according to the assumptions (6.1) and (6.2), we obtain

$$\begin{aligned} \Theta(\tilde{\gamma} \cdot (\Omega, Z)) &= \langle \theta, \mathcal{F}_{\tilde{\gamma} \cdot (\Omega, Z)} \rangle \\ &= \langle \theta, \psi(\tilde{\gamma})^{-1} J(\tilde{\gamma}, (\Omega, Z)) \pi(\tilde{\gamma}) \mathcal{F}_{\Omega, Z} \rangle \\ &= \psi(\tilde{\gamma})^{-1} J(\tilde{\gamma}, (\Omega, Z)) \langle \theta, \pi(\tilde{\gamma}) \mathcal{F}_{\Omega, Z} \rangle \\ &= \psi(\tilde{\gamma})^{-1} \chi(\tilde{\gamma})^{-1} J(\tilde{\gamma}, (\Omega, Z)) \langle \theta, \mathcal{F}_{\Omega, Z} \rangle \\ &= \psi(\tilde{\gamma})^{-1} \chi(\tilde{\gamma})^{-1} J(\tilde{\gamma}, (\Omega, Z)) \Theta(\Omega, Z). \end{aligned}$$

□

Now for a positive definite integral symmetric matrix \mathcal{M} of degree m , we define the holomorphic function $\Theta_{\mathcal{M}} : \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ by

$$(6.5) \quad \Theta_{\mathcal{M}}(\Omega, Z) := \sum_{\xi \in \mathbb{Z}^{(m,n)}} e^{\pi i \sigma(\mathcal{M}(\xi \Omega^t \xi + 2\xi^t Z))}, \quad (\Omega, Z) \in \mathbb{H}_{n,m}.$$

Theorem 6.1. *Let \mathcal{M} be a symmetric positive definite, unimodular even integral matrix of degree m . Then for any $\tilde{\gamma} = (\gamma, (\lambda, \mu; \kappa)) \in \Gamma^J$ with $\gamma \in \Gamma_n$ and $(\lambda, \mu; \kappa) \in H_{\mathbb{Z}}^{(n,m)}$, the function $\Theta_{\mathcal{M}}$ satisfies the functional equation*

$$(6.6) \quad \Theta_{\mathcal{M}}(\tilde{\gamma} \cdot (\Omega, Z)) = \rho_{\mathcal{M}}(\tilde{\gamma}) J_{\mathcal{M}}(\tilde{\gamma}, (\Omega, Z)) \Theta_{\mathcal{M}}(\Omega, Z), \quad (\Omega, Z) \in \mathbb{H}_{n,m},$$

where $\rho_{\mathcal{M}}(\tilde{\gamma})$ is a uniquely determined character of Γ^J with $|\rho_{\mathcal{M}}(\tilde{\gamma})|^8 = 1$ and $J_{\mathcal{M}} : G^J \times \mathbb{H}_{n,m} \longrightarrow \mathbb{C}^\times$ is the automorphic factor for G^J on $\mathbb{H}_{n,m}$ defined by the formula (5.2).

Proof. For an element $\tilde{\gamma} = (\gamma, (\lambda, \mu; \kappa)) \in \Gamma^J$ with $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$ and $(\lambda, \mu; \kappa) \in H_{\mathbb{Z}}^{(n,m)}$, we put $(\Omega_*, Z_*) = \tilde{\gamma} \cdot (\Omega, Z)$ for $(\Omega, Z) \in \mathbb{H}_{n,m}$. Then we have

$$\begin{aligned} \Omega_* &= \gamma \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \\ Z_* &= (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1}. \end{aligned}$$

We define the linear functional ϑ on $L^2(\mathbb{R}^{(m,n)})$ by

$$\vartheta(f) = \langle \vartheta, f \rangle := \sum_{\xi \in \mathbb{Z}^{(m,n)}} f(\xi), \quad f \in L^2(\mathbb{R}^{(m,n)}).$$

We note that $\Theta_{\mathcal{M}}(\Omega, Z) = \vartheta(\mathcal{F}_{\Omega, Z}^{(\mathcal{M})})$. Since $\mathcal{F}^{(\mathcal{M})}$ is a covariant map for the Schrödinger-Weil representation $\omega_{\mathcal{M}}$ by Theorem 5.1, according to Lemma 6.1, it suffices to prove that ϑ is semi-invariant for $\omega_{\mathcal{M}}$ under the action of Γ^J , in other words, ϑ satisfies the following semi-invariance relation

$$(6.7) \quad \left\langle \vartheta, \omega_{\mathcal{M}}(\tilde{\gamma}) \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \right\rangle = \rho_{\mathcal{M}}(\tilde{\gamma})^{-1} \left\langle \vartheta, \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \right\rangle$$

for all $\tilde{\gamma} \in \Gamma^J$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$.

We see that the following elements $h(\lambda, \mu; \kappa)$, $t(b)$, $g(\alpha)$ and σ_n of Γ^J defined by

$$\begin{aligned} h(\lambda, \mu; \kappa) &= (I_{2n}, (\lambda, \mu; \kappa)) \text{ with } \lambda, \mu \in \mathbb{Z}^{(m,n)} \text{ and } \kappa \in \mathbb{Z}^{(m,m)}, \\ t(b) &= (t_0(b), (0, 0; 0)) \text{ with any } b = {}^t b \in \mathbb{Z}^{(n,n)}, \\ g(\alpha) &= (g_0(\alpha), (0, 0; 0)) \text{ with any } \alpha \in GL(n, \mathbb{Z}), \\ \sigma_n &= (s_{n,0}, (0, 0; 0)) \end{aligned}$$

generate the Jacobi modular group Γ^J . Therefore it suffices to prove the semi-invariance relation (6.7) for the above generators of Γ^J .

Case I. $\tilde{\gamma} = h(\lambda, \mu; \kappa)$ with $\lambda, \mu \in \mathbb{Z}^{(m,n)}$, $\kappa \in \mathbb{Z}^{(m,m)}$.

In this case, we have

$$\Omega_* = \Omega, \quad Z_* = Z + \lambda \Omega + \mu$$

and

$$J_{\mathcal{M}}(\tilde{\gamma}, (\Omega, Z)) = e^{-\pi i \sigma \{ \mathcal{M}(\lambda \Omega^t \lambda + 2 \lambda^t Z + \kappa + \mu^t \lambda) \}}.$$

According to the covariance relation (5.3),

$$\begin{aligned} & \left\langle \vartheta, \omega_{\mathcal{M}}(\tilde{\gamma}) \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \right\rangle \\ &= \left\langle \vartheta, J_{\mathcal{M}}(\tilde{\gamma}, (\Omega, Z))^{-1} \mathcal{F}_{\tilde{\gamma} \cdot (\Omega, Z)}^{(\mathcal{M})} \right\rangle \\ &= J_{\mathcal{M}}(\tilde{\gamma}, (\Omega, Z))^{-1} \left\langle \vartheta, \mathcal{F}_{\Omega, Z + \lambda \Omega + \mu}^{(\mathcal{M})} \right\rangle \\ &= J_{\mathcal{M}}(\tilde{\gamma}, (\Omega, Z))^{-1} \sum_{A \in \mathbb{Z}^{(m,n)}} e^{\pi i \sigma \{ \mathcal{M}(A \Omega^t A + 2 A^t (Z + \lambda \Omega + \mu)) \}} \\ &= J_{\mathcal{M}}(\tilde{\gamma}, (\Omega, Z))^{-1} \cdot e^{-\pi i \sigma(\mathcal{M}(\lambda \Omega^t \lambda + 2 \lambda^t Z))} \\ & \quad \times \sum_{A \in \mathbb{Z}^{(m,n)}} e^{2 \pi i \sigma(\mathcal{M} A^t \mu)} e^{\pi i \sigma \{ \mathcal{M}((A + \lambda) \Omega^t (A + \lambda) + 2 (A + \lambda)^t Z) \}} \\ &= e^{\pi i \sigma(\mathcal{M}(\kappa + \mu^t \lambda))} \left\langle \vartheta, \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \right\rangle. \end{aligned}$$

Here we used the fact that $\sigma(\mathcal{M} A^t \mu)$ is an integer. We put $\rho_{\mathcal{M}}(\tilde{\gamma}) = \rho_{\mathcal{M}}(h(\lambda, \mu; \kappa)) = e^{-\pi i \sigma(\mathcal{M}(\kappa + \mu^t \lambda))}$. Therefore ϑ satisfies the semi-invariance relation (6.7) in the case $\tilde{\gamma} = h(\lambda, \mu; \kappa)$ with $\lambda, \mu \in \mathbb{Z}^{(m,n)}$, $\kappa \in \mathbb{Z}^{(m,m)}$.

Case II. $\tilde{\gamma} = t(b)$ with $b = {}^t b \in \mathbb{Z}^{(n,n)}$.

In this case, we have

$$\Omega_* = \Omega + b, \quad Z_* = Z \quad \text{and} \quad J_{\mathcal{M}}(\tilde{\gamma}, (\Omega, Z)) = 1.$$

According to the covariance relation (5.3), we obtain

$$\begin{aligned} & \langle \vartheta, \omega_{\mathcal{M}}(\tilde{\gamma}) \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \rangle \\ &= \langle \vartheta, J_{\mathcal{M}}(\tilde{\gamma}, (\Omega, Z))^{-1} \mathcal{F}_{\tilde{\gamma} \cdot (\Omega, Z)}^{(\mathcal{M})} \rangle \\ &= \langle \vartheta, \mathcal{F}_{\Omega+b, Z}^{(\mathcal{M})} \rangle \\ &= \sum_{A \in \mathbb{Z}^{(m,n)}} e^{\pi i \sigma \{ \mathcal{M}(A(\Omega+b)^t A + 2 A^t Z) \}} \\ &= \sum_{A \in \mathbb{Z}^{(m,n)}} e^{\pi i \sigma(\mathcal{M}(A \Omega^t A + 2 A^t Z))} \cdot e^{\pi i \sigma(\mathcal{M} A b^t A)} \\ &= \sum_{A \in \mathbb{Z}^{(m,n)}} e^{\pi i \sigma(\mathcal{M}(A \Omega^t A + 2 A^t Z))} \\ &= \langle \vartheta, \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \rangle. \end{aligned}$$

Here we used the fact that $\sigma(\mathcal{M} A b^t A)$ is an even integer. We put $\rho_{\mathcal{M}}(\tilde{\gamma}) = \rho_{\mathcal{M}}(t(b)) = 1$. Therefore ϑ satisfies the semi-invariance relation (6.7) in the case $\tilde{\gamma} = t(b)$ with $b = {}^t b \in \mathbb{Z}^{(n,n)}$.

Case III. $\tilde{\gamma} = g(\alpha)$ with $\alpha \in GL(n, \mathbb{Z})$.

In this case, we have

$$\Omega_* = {}^t \alpha \Omega \alpha, \quad Z_* = Z \alpha$$

and

$$J_{\mathcal{M}}(\tilde{\gamma}, (\Omega, Z)) = (\det \alpha)^{-\frac{m}{2}}.$$

According to the covariance relation (5.3), we obtain

$$\begin{aligned} & \langle \vartheta, \omega_{\mathcal{M}}(\tilde{\gamma}) \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \rangle \\ &= \langle \vartheta, J_{\mathcal{M}}(\tilde{\gamma}, (\Omega, Z))^{-1} \mathcal{F}_{\tilde{\gamma} \cdot (\Omega, Z)}^{(\mathcal{M})} \rangle \\ &= (\det \alpha)^{\frac{m}{2}} \langle \vartheta, \mathcal{F}_{t \alpha \Omega \alpha, Z \alpha}^{(\mathcal{M})} \rangle \\ &= (\det \alpha)^{\frac{m}{2}} \sum_{A \in \mathbb{Z}^{(m,n)}} \mathcal{F}_{t \alpha \Omega \alpha, Z \alpha}^{(\mathcal{M})}(A) \\ &= (\det \alpha)^{\frac{m}{2}} \sum_{A \in \mathbb{Z}^{(m,n)}} e^{\pi i \sigma \{ \mathcal{M}(A^t \alpha \Omega^t (A^t \alpha) + 2 A^t \alpha^t Z) \}} \\ &= (\det \alpha)^{\frac{m}{2}} \langle \vartheta, \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \rangle. \end{aligned}$$

Here we put $\rho_{\mathcal{M}}(\tilde{\gamma}) = \rho_{\mathcal{M}}(g(\alpha)) = (\det \alpha)^{-\frac{m}{2}}$. Therefore ϑ satisfies the semi-invariance relation (6.7) in the case $\tilde{\gamma} = g(\alpha)$ with $\alpha \in GL(n, \mathbb{Z})$.

Case IV. $\tilde{\gamma} = \left(\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, (0, 0; 0) \right)$.

In this case, we have

$$\Omega_* = -\Omega^{-1}, \quad Z_* = Z \Omega^{-1}$$

and

$$J_{\mathcal{M}}(\tilde{\gamma}, (\Omega, Z)) = e^{\pi i \sigma(\mathcal{M} Z \Omega^{-1 t} Z)} (\det \Omega)^{\frac{m}{2}}.$$

In the process of the proof of Theorem 5.1, using Lemma 5.1, we already showed that

$$(6.8) \quad \int_{\mathbb{R}^{(m,n)}} e^{\pi i \sigma(\mathcal{M}(y \Omega^t y + 2 y^t Z))} dy = (\det \mathcal{M})^{-\frac{n}{2}} \left(\det \frac{\Omega}{i} \right)^{-\frac{m}{2}} e^{-\pi i \sigma(\mathcal{M} Z \Omega^{-1 t} Z)}.$$

By (6.8), we see that the Fourier transform of $\mathcal{F}_{\Omega, Z}^{(\mathcal{M})}$ is given by

$$(6.9) \quad \widehat{\mathcal{F}_{\Omega, Z}^{(\mathcal{M})}}(x) = (\det \mathcal{M})^{-\frac{n}{2}} \left(\det \frac{\Omega}{i} \right)^{-\frac{m}{2}} e^{-\pi i \sigma(\mathcal{M}(Z-x) \Omega^{-1 t}(Z-x))}.$$

According to the covariance relation (5.3), Formula (6.9) and Poisson summation formula, we obtain

$$\begin{aligned} & \langle \vartheta, \omega_{\mathcal{M}}(\tilde{\gamma}) \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \rangle \\ &= \langle \vartheta, J_{\mathcal{M}}(\tilde{\gamma}, (\Omega, Z))^{-1} \mathcal{F}_{\tilde{\gamma}, (\Omega, Z)}^{(\mathcal{M})} \rangle \\ &= J_{\mathcal{M}}(\tilde{\gamma}, (\Omega, Z))^{-1} \langle \vartheta, \mathcal{F}_{-\Omega^{-1}, Z \Omega^{-1}}^{(\mathcal{M})} \rangle \\ &= (\det \Omega)^{-\frac{m}{2}} e^{-\pi i \sigma(\mathcal{M} Z \Omega^{-1 t} Z)} \sum_{A \in \mathbb{Z}^{(m,n)}} e^{-\pi i \sigma(\mathcal{M}(A \Omega^{-1 t} A - 2 A \Omega^{-1 t} Z))} \\ &= (\det \Omega)^{-\frac{m}{2}} \sum_{A \in \mathbb{Z}^{(m,n)}} e^{-\pi i \sigma(\mathcal{M}(Z \Omega^{-1 t} Z + A \Omega^{-1 t} A - 2 A \Omega^{-1 t} Z))} \\ &= (\det \Omega)^{-\frac{m}{2}} \sum_{A \in \mathbb{Z}^{(m,n)}} e^{-\pi i \sigma(\mathcal{M}(Z-A) \Omega^{-1 t}(Z-A))} \\ &= (\det \Omega)^{-\frac{m}{2}} (\det \mathcal{M})^{\frac{n}{2}} \left(\det \frac{\Omega}{i} \right)^{\frac{m}{2}} \sum_{A \in \mathbb{Z}^{(m,n)}} \widehat{\mathcal{F}_{\Omega, Z}^{(\mathcal{M})}}(A) \quad (\text{by Formula (6.9)}) \\ &= (\det \mathcal{M})^{\frac{n}{2}} \left(\det \frac{I_n}{i} \right)^{\frac{m}{2}} \sum_{A \in \mathbb{Z}^{(m,n)}} \mathcal{F}_{\Omega, Z}^{(\mathcal{M})}(A) \quad (\text{by Poisson summation formula}) \\ &= (\det \mathcal{M})^{\frac{n}{2}} (-i)^{\frac{mn}{2}} \langle \vartheta, \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \rangle \\ &= (-i)^{\frac{mn}{2}} \langle \vartheta, \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \rangle. \end{aligned}$$

Here we used the fact that $\det \mathcal{M} = 1$ because \mathcal{M} is unimodular. We put $\rho_{\mathcal{M}}(\tilde{\gamma}) = \rho_{\mathcal{M}}(\sigma_n) = (-i)^{-\frac{mn}{2}}$. Therefore ϑ satisfies the semi-invariance relation (6.7) in the case $\tilde{\gamma} = \sigma_n$. The proof of Case IV is completed. Since $J_{\mathcal{M}}$ is an automorphic factor for G^J on $\mathbb{H}_{n,m}$, we see that if the formula (6.6) holds for two elements $\tilde{\gamma}_1, \tilde{\gamma}_2$ in Γ^J , then it holds for $\tilde{\gamma}_1 \tilde{\gamma}_2$. Finally we complete the proof of Theorem 6.1. \square

Remark 6.1. For a symmetric positive definite integral matrix \mathcal{M} that is not unimodular even integral, we obtain a similar transformation formula like (6.6). If m is odd, $\Theta_{\mathcal{M}}(\Omega, Z)$ is a Jacobi form of a half-integral weight $\frac{m}{2}$ and index $\frac{\mathcal{M}}{2}$ with respect to a suitable arithmetic subgroup $\Gamma_{\Theta, \mathcal{M}}^J$ of Γ^J and a character $\rho_{\mathcal{M}}$ of $\Gamma_{\Theta, \mathcal{M}}^J$.

For instance, we obtain the following :

Theorem 6.2. Let \mathcal{M} be a symmetric positive definite integral matrix of degree m such that $\det(\mathcal{M}) = 1$. Let $\Gamma_{1,2}$ be an arithmetic subgroup of Γ_n generated by all the following elements

$$t(b) = \begin{pmatrix} I_n & b \\ 0 & I_n \end{pmatrix}, \quad g(\alpha) = \begin{pmatrix} {}^t \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad \sigma_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

where $b = {}^t b \in \mathbb{Z}^{(n,n)}$ with even diagonal and $\alpha \in \mathbb{Z}^{(n,n)}$. We put

$$\Gamma_{1,2}^J := \Gamma_{1,2} \ltimes H_{\mathbb{Z}}^{(n,m)}.$$

Then $\Theta_{\mathcal{M}}$ satisfies the transformation formula (6.6) for all $\tilde{\gamma} \in \Gamma_{1,2}^J$. Therefore $\Theta_{\mathcal{M}}$ is a Jacobi form of weight $\frac{m}{2}$ with level $\Gamma_{1,2}$ and index $\frac{\mathcal{M}}{2}$ for the uniquely determined character $\rho_{\mathcal{M}}$ of $\Gamma_{1,2}^J$.

Proof. The proof is essentially the same as the proof of Theorem 6.1. We leave the detail to the reader. \square

REFERENCES

- [1] R. Berndt and R. Schmidt, *Elements of the Representation Theory of the Jacobi Group*, Birkhäuser, 1998.
- [2] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Progress in Math., **55**, Birkhäuser, Boston, Basel and Stuttgart, 1985.
- [3] E. Freitag, *Siegelsche Modulfunktionen*, Grundlehren der mathematischen Wissenschaften **55**, Springer-Verlag, Berlin-Heidelberg-New York (1983).
- [4] E. Hecke, *Herleitung des Euler-Produktes der Zetafunktion und einiger L-Reihen aus ihrer Funktionalgleichung*, Math. Ann. **119** (1944), 266-287 (=Werke, 919-940).
- [5] S. Gelbart, *Weil's Representation and the Spectrum of the Metaplectic Group*, Lecture Notes in Math. **530**, Springer-Verlag, Berlin and New York, 1976.
- [6] M. Kashiwara and M. Vergne, *On the Segal-Shale-Weil Representations and Harmonic Polynomials*, Invent. Math. **44** (1978), 1-47.
- [7] G. Lion and M. Vergne, *The Weil representation, Maslov index and Theta series*, Progress in Math., **6**, Birkhäuser, Boston, Basel and Stuttgart, 1980.
- [8] G. W. Mackey, *Induced Representations of Locally Compact Groups I*, Ann. of Math., **55** (1952), 101-139.
- [9] J. Marklof, *Pair correlation densities of inhomogeneous quadratic forms*, Ann. of Math., **158** (2003), 419-471.
- [10] D. Mumford, *Tata Lectures on Theta I*, Progress in Math. **28**, Boston-Basel-Stuttgart (1983).
- [11] G. Shimura, *On modular forms of half integral weight*, Ann. of Math., **97** (1973), 440-481; Collected Papers, 1967-1977, Vol. II, Springer-Verlag (2002), 532-573.
- [12] C. L. Siegel, *Indefinite quadratische Formen und Funktionentheorie I and II*, Math. Ann. **124** (1951), 17-54 and Math. Ann. **124** (1952), 364-387; Gesammelte Abhandlungen, Band III, Springer-Verlag (1966), 105-142 and 154-177.

- [13] A. Weil, *Sur certains groupes d'opérateurs unitaires*, Acta Math., **111** (1964), 143–211; Collected Papers (1964–1978), Vol. III, Springer-Verlag (1979), 1–69.
- [14] J.-H. Yang, *Harmonic Analysis on the Quotient Spaces of Heisenberg Groups*, Nagoya Math. J., **123** (1991), 103–117.
- [15] J.-H. Yang, *Harmonic Analysis on the Quotient Spaces of Heisenberg Groups II*, J. Number Theory, **49** (1) (1994), 63–72.
- [16] J.-H. Yang, *A decomposition theorem on differential polynomials of theta functions of high level*, Japanese J. of Mathematics, the Mathematical Society of Japan, New Series, **22** (1) (1996), 37–49.
- [17] J.-H. Yang, *Fock Representations of the Heisenberg Group $H_{\mathbb{R}}^{(g,h)}$* , J. Korean Math. Soc., **34**, no. 2 (1997), 345–370.
- [18] J.-H. Yang, *Lattice Representations of the Heisenberg Group $H_{\mathbb{R}}^{(g,h)}$* , Math. Annalen, **317** (2000), 309–323.
- [19] J.-H. Yang, *The Siegel-Jacobi Operator*, Abh. Math. Sem. Univ. Hamburg **63** (1993), 135–146.
- [20] J.-H. Yang, *Remarks on Jacobi forms of higher degree*, Proc. of the 1993 Workshop on Automorphic Forms and Related Topics, the Pyungsan Institute for Mathematical Sciences, Seoul (1993), 33–58.
- [21] J.-H. Yang, *Singular Jacobi forms*, Trans. of American Math. Soc. **347**, No. 6 (1995), 2041–2049.
- [22] J.-H. Yang, *Construction of vector valued modular forms from Jacobi forms*, Canadian J. of Math. **47** (6) (1995), 1329–1339.
- [23] J.-H. Yang, *A note on a fundamental domain for Siegel-Jacobi space*, Houston Journal of Mathematics, Vol. **32**, No. 3 (2006), 701–712.
- [24] J.-H. Yang, *Invariant metrics and Laplacians on Siegel-Jacobi space*, Journal of Number Theory, **127** (2007), 83–102 or arXiv:math.NT/0507215.
- [25] J.-H. Yang, *A partial Cayley transform of Siegel-Jacobi disk*, J. Korean Math. Soc. **45**, No. 3 (2008), 781–794.
- [26] C. Ziegler, *Jacobi Forms of Higher Degree*, Abh. Math. Sem. Hamburg **59** (1989), 191–224.

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